Sequences

Definition 15
Let $X$ be any nonempty set. A sequence in $X$ is a function $f: \mathbb{N} \to X$.

In the above, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

It is customary to write $x_n$ in place of $f(n)$ and to call $x_n$ the $n^{th}$ term of the sequence.

We also write $f$ as $\{x_1, x_2, x_3, \ldots\}$ or, more compactly, $\{x_n\}_{n=1}^{\infty}$. If we’re feeling particularly lazy, we may just write $f$ as $\{x_n\}$. It is convenient to allow sequences to be any function whose domain is $\mathbb{N}$ or a subset of $\mathbb{N}$. That is, in some cases it may be convenient to index a sequence by the even natural numbers or the primes.

Example 7

1. Let $X = \mathbb{Q}$ and define $f(n) = \frac{1}{n^2}$. Then $f = \{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ is a sequence in $\mathbb{Q}$.

Being a bit more explicit, $f = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\right\}$. We observe the that the terms $x_n$ of the sequence get “closer and closer” to zero as the value of the input $n$ gets larger and larger.

2. Let $X = \mathbb{Q}$ and define $f = \{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots\}$. Here we observe that in particular $x_3 = 1.41$ and $x_{15} = 1.41421356237309$ and that as the index $n$ increases the corresponding values of $x_n$ seemingly “approach” $\sqrt{2}$, which is not an element of the set of rational numbers.

3. Let $X = \mathbb{Q}$ and define $\{x_n\}_{n=1}^{\infty} = \left\{(-1)^n n\right\}_{n=1}^{\infty}$. It is clear that the range of this sequence is (doubly!) unbounded and, hence, does not approach any given real number as the index $n$ increases without bound.
4. Let $X$ be the set of all continuous functions defined on $\mathbb{R}$. Then $f: \mathbb{N} \to X$ defined by

$$f(n) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

is a sequence in $X$ where $f_n$ is the $n^{th}$ Taylor polynomial for the analytic function $y = e^x$. More explicitly, the sequence is the set of differentiable functions given by

$$\left\{ 1 + x + \frac{x^2}{2}, 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \ldots \right\}.$$

What might it mean here to say that the above sequence converges?

5. Let $X$ be the set of rational numbers and define the $n^{th}$ term of the sequence $(n \geq 2)$ by

$$x_n = \prod_{k=2}^{n} \left( 1 - \frac{1}{k^2} \right)$$

$$= \left( 1 - \frac{1}{2^2} \right) \left( 1 - \frac{1}{3^2} \right) \left( 1 - \frac{1}{4^2} \right) \ldots \left( 1 - \frac{1}{n^2} \right).$$

$$= x_{n-1} \left( 1 - \frac{1}{n^2} \right)$$

Then we have that $x_3 = \frac{3}{5}, x_{15} = \frac{8}{15}, x_{167} = \frac{84}{167}, \& x_{12996} = \frac{12997}{25992}$ Based on the extremely limited data shown here one might conjecture that the terms of this sequence approach $\frac{1}{2}$. In fact, this is indeed the case. That is $\prod_{k=2}^{n} \left( 1 - \frac{1}{k^2} \right) = \frac{1}{2}$. A simple mathematical induction argument shows that $x_n = \frac{n + 1}{2n} - \frac{1}{2}$.

6. Define a sequence as follows:

$$x_1 = \{ 1, 0, 0, 0, 0, \ldots \}$$

$$x_2 = \{ 0, 1, 0, 0, 0, \ldots \}$$
Does the above sequence “converge” in some sense?

Observe that a sequence may consist of a set of numbers (parts 1-3 & 5 above) or some set of objects such functions (items 4 & 6 above) or, quite frankly, any interesting mathematical object. The notion of a sequence “approaching” a value (i.e., converging to a limit) is important and will be discussed later. (Just what might it mean for a the sequences in parts 4 & 6 above to converge?)

Illustration 1

1. Let \( f: \mathbb{N} \to \mathbb{Q} \) defined by \( f(n) = x_n = \begin{cases} \frac{n}{n^2 + 1} & \text{if } n \text{ is odd} \\ \frac{n}{3^n} & \text{if } n \text{ is even} \end{cases} \). In this case, \( \{x_n\}_{n=1}^{\infty} \) is a sequence “approaching” zero.

2. Let \( f: \mathbb{N} \to \mathbb{Q} \) defined by \( f(n) = x_n = \sum_{k=0}^{n} \left( \frac{1}{5} \right)^k \). Then \( \{x_n\}_{n=1}^{\infty} \) is a sequence approaching the value \( \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} \). (See geometric series in any calculus text.)

3. Let \( x_1 = 1 \) and \( x_n = \frac{1}{2} \left( x_{n-1} + \frac{2}{x_{n-1}} \right) \) for \( n \geq 2 \). Then

\[
\{x_n\}_{n=1}^{\infty} = \left\{ 1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{66557}{470832}, \ldots \right\} = \left\{ 1, 1.5, 1.416, 1.41421568627450980392, \ldots \right\}
\]
is a sequence of rational numbers approaching $\sqrt{2}$. (Solve the equation $\alpha = \frac{1}{2} \left( \alpha + \frac{2}{\alpha} \right)$ for $\alpha$. Does your solution relate to the “limit value” for the sequence?)

4. Let $X$ be the set of rational numbers and define the $n^{th}$ term of the sequence $\{x_n\}^{n \geq 2}$ by

$$x_n = \prod_{k=2}^{n} \left( 1 - \frac{2}{k^3 + 1} \right).$$

It is possible to show that $x_n \to \frac{2}{3}$. Verify this statement by computing a few terms of this sequence.

We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is increasing (respectively, decreasing) iff $x_1 \leq x_2 \leq x_3 \leq \ldots$ (respectively, $x_1 \geq x_2 \geq x_3 \geq \ldots$). Further, we say that a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is bounded provided that there exists a real number $M > 0$ so that $|x_n| < M$ for all indices $n \in \mathbb{N}$. A sequence that is not bounded is said to be unbounded. Classify parts (1)-(3), & (5) of Example 7 as increasing, decreasing, or neither and bounded or unbounded.

**Illustration 2**

1. The sequence $x_n = \frac{1}{n}$ is bounded between 0 and 1.

2. The recursive sequence given by $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{1}{2 + x_n}$ is bounded between $\frac{1}{3} \& \frac{1}{2}$.

3. The sequence $\left\{0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 1, \frac{1}{4}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, 0, \ldots \right\}$ is bounded between 0 and 1 inclusive.
We now define the addition and multiplication of sequences. Both of these operations are
defined via term-by-term addition/multiplication of the elements.

**Definition 16**

Let $X$ be a set with two binary operations, say $+$ and $\cdot$ (hereafter denoted by jux-a-position). Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in $X$.

1. The sum of $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$, denoted by $\{x_n\}_{n=1}^\infty + \{y_n\}_{n=1}^\infty$, is the sequence in $X$
given by $\{x_n + y_n\}_{n=1}^\infty$.

2. The product of $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$, denoted by $\{x_n\}_{n=1}^\infty \cdot \{y_n\}_{n=1}^\infty$, is the sequence in $X$
given by $\{x_n \cdot y_n\}_{n=1}^\infty$.

The algebraic structure of $(X, +, \cdot)$ need not be a nice as a field. In fact, strictly speaking, for
Definition 16 to be well-defined it is enough that $X$ simply have defined on its elements the
operations of addition and multiplication. Of course, it should come as no surprise that the more
structure (associative properties, commutative properties, etc.) that $(X, +, \cdot)$ has associated with it, the
more “corresponding” structure the sum and product of its sequences in $X$ will possess.

While the above notation is standard mathematical notation, it does lead to some potential
confusion as the addition and multiplication symbols have two meanings; one meaning is connected
with the algebraic system in $X$ while the other has to do with the addition and multiplication of
sequences in $X$.

**Example 8**

1. Let $\left\{\frac{1}{n}\right\}_{n=1}^\infty$ and $\left\{1 - \frac{1}{n^3}\right\}_{n=1}^\infty$ be sequences in $\mathbb{Q}$. Then
$$\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} + \left\{ 1 - \frac{1}{n^3} \right\}_{n=1}^{\infty} = \left\{ \frac{n^3 + n^2 - 1}{n^3} \right\}_{n=1}^{\infty} = \left\{ 1 + \frac{1}{n} - \frac{1}{n^3} \right\}_{n=1}^{\infty}$$

and

$$\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \left( 1 - \frac{1}{n^3} \right)_{n=1}^{\infty} = \left\{ \frac{n^3 - 1}{n^4} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{n} - \frac{1}{n^4} \right\}_{n=1}^{\infty}.$$  

(Do these sequences approach a limiting value as $n$ approaches $\infty$?)

2. Let $\left\{ x^2 e^{-nx} \right\}_{n=1}^{\infty}$ and $\left\{ \frac{x^n}{1 + x^n} \right\}_{n=1}^{\infty}$ be sequences in the space of functions continuous on $[0, \infty)$. Form the sum and product of these sequences. (Do the resulting sequences approach a limiting value as $n$ approaches $\infty$?)

Example 9

1. Let $x_n = \frac{1}{n}$ ($n \in \mathbb{N}$). Experience suggests that $x_n \to 0$ as $n \to \infty$. That is, the $n^{th}$ term of the sequence gets closer and closer to 0 as the indices get larger and larger. Stated another way, the distance between $\frac{1}{n}$ and 0 gets smaller and smaller as the value of $n$ gets bigger and bigger. Then $\left| \frac{1}{n} - 0 \right|$ measures the distance between $\frac{1}{n}$ and 0. Show that we can find values of $N$ so that $\left| \frac{1}{n} - 0 \right| < \varepsilon$ ($\varepsilon = 1, 0.1, 0.0305$) for all values of the index $n$ greater than or equal to $N$.

2. Let $x_n = \sqrt{n} + 1 - \sqrt{n}$. Methods from calculus show that the given sequence approaches
zero as \( n \) approaches \( \infty \). Show that we can find values of \( N \) so that
\[
\left| \frac{1}{n} - 0 \right| < \varepsilon
\]
for all values of the index \( n \) greater than or equal to \( N \).

Hint: \( |x_n - 0| < \frac{1}{2\sqrt{n}} \).

**Definition 17**

A sequence \( \{x_n\}_{n=1}^\infty \) in \( \mathbb{R} \) is said to **converge** to (a **limit value**) \( L \) iff for each \( \varepsilon > 0 \) there exists a natural number \( N \) so that if \( n \geq N \), \( |x_n - L| < \varepsilon \). In this case we write \( x_n \to L \).

A sequence that *fails* to converge is said to **diverge**. Of course, \( x_n \to L \) may also be written as \( \lim_{n \to \infty} x_n = L \). What would you need to know to well-define the convergence of a sequence of, say, continuous functions on \( \mathbb{R} \)?

**Theorem 18**

If \( \{x_n\}_{n=1}^\infty \) is a convergent sequence, then its limit value is unique.

**Proof - Optional**

On the contrary, suppose that \( x_n \to L \) and \( x_n \to M \) where \( L < M \). Set \( \varepsilon = \frac{M - L}{2} \). Then there exist \( N_L \) so that if \( n \geq N_L \), we have that \( |x_n - L| < \varepsilon \) and there exist \( N_M \) so that if \( n \geq N_M \), we have that \( |x_n - M| < \varepsilon \). Set \( N = \max \{ N_L, N_M \} \). Then for any \( n \geq N \) we have that \( x_n < L + \varepsilon < M - \varepsilon < x_n \). A contradiction. We conclude that sequences of real numbers converge to a unique limiting value. \(
\)
Example 10

1. Show that \( \frac{1}{n} \to 0 \)

2. Prove that \( x_n = \sqrt{n + 1} - \sqrt{n} \to 0 \)

3. Let \( x_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \). Then show that \( x_n \to 0 \).

4. Prove that \( x_n = \frac{4n^2 + 6}{n^4 + 3n} \) converges.

Solution

Suppose \( \varepsilon > 0 \). Claim: \( \frac{4n^2 + 6}{n^4 + 3n} \to 0 \). Then for all natural numbers \( n \) we have that

\[
\left| \frac{4n^2 + 6}{n^4 + 3n} - 0 \right| = \frac{4 + \frac{6}{n^2}}{n^2 + \frac{3}{n}} \leq \frac{4 + \frac{6}{n^2}}{n^2 + \frac{3}{n}} \leq \frac{10}{n^2}.
\]

Choose \( N \in \mathbb{N} \) such that \( N > \sqrt{\frac{\varepsilon}{10}} \). Then for \( n \geq N \) we have that \( \frac{10}{n^2} \leq \frac{10}{N^2} < \varepsilon \). That is,

for \( n \geq N \) we have that \( \left| \frac{4n^2 + 6}{n^4 + 3n} - 0 \right| < \varepsilon \) and so \( \frac{4n^2 + 6}{n^4 + 3n} \to 0 \).
5. Discuss the convergence of \( x_n = 1 + r^n \) for some \( r > 0 \).

\( \text{(Cases: } 0 < r < 1, r = 1, r > 1 \text{ )} \)

6. Let \( x_n = \frac{n}{\sqrt{n}} \). Then prove that \( x_n \to 1 \).

**Solution**

Set \( y_n = \frac{n}{\sqrt{n}} - 1 \). Then \( y_n \geq 0 \) and by the *Binomial Theorem* (e.g. see Lemma 7.1 page 180 Protter & Morrey) we have

\[
\begin{align*}
n &= (1 + y_n)^n \\
&\geq n \frac{n(n-1)}{2} y_n^2 \\
&\Rightarrow \frac{2}{n(n-1)} \geq y_n^2 \\
&\Rightarrow \sqrt{\frac{2}{n-1}} \geq y_n > 0.
\end{align*}
\]

Since \( \sqrt{\frac{2}{n-1}} \to 0 \) as \( n \to \infty \), we conclude that \( y_n \to 0 \). (Think *Sandwich or Pinching Theorem* from ordinary calculus.) The desired result follows.

7. Let \( x_n = 1 + (-1)^n \). Then the sequence \( \{x_n\}_{n=1}^{\infty} = \{0, 1, 0, 1 ... \} \) fails to converge.

8. Let \( x_n = \frac{n^3 + n + 17}{n} \). Then the sequence \( \{x_n\}_{n=1}^{\infty} \) fails to converge.

**Illustration 3**

Prove that if \( x_n \to 0 \) and if \( y_n \) is a bounded sequence, then \( x_n y_n \to 0 \).

**Proof**

Let \( \varepsilon > 0 \). Find \( N \) so that if \( n > N \) we have that \( \left| x_n y_n - 0 \right| < \varepsilon \). Since \( y_n \) is a bounded
sequence, there exists a real number \( M > 0 \) so that \( |y_n| < M \). Since \( x_n \to 0 \) there exists \( N \) so that if \( n > N \) we have that \( |x_n - 0| < \frac{\varepsilon}{M} \). Now, for \( n > N \) we have that

\[
|x_n y_n - 0| < |y_n| |x_n - 0| < M |x_n - 0| < M \frac{\varepsilon}{M} < \varepsilon.
\]

Thus, \( x_n y_n \to 0 \). \( \star \)

**Proposition 19**

Every convergent sequence \( \{x_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) is bounded.

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**Proof - Optional**

Suppose \( x_n \to x \) and let \( \varepsilon = 1 \). Then there exists a natural number \( N \) so that \( |x_n - x| < \varepsilon = 1 \) for all \( n \geq N \). That is, \( -1 < x_n - x < 1 \) for \( n \geq N \) and so \( x - 1 < x_n < x + 1 \) for \( n \geq N \).

Now, set

\[
L = \min \{ x_1, x_2, x_3, \ldots, x_{n-1}, x - 1 \}
\]

and

\[
M = \max \{ x_1, x_2, x_3, \ldots, x_{n-1}, x + 1 \}.
\]

Then for all natural numbers \( n \) we have that \( L \leq x_n \leq M \) and we conclude \( \{x_n\}_{n=1}^{\infty} \) is bounded. \( \star \)

We note that the sequence \( \left\{ 0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \ldots \right\} \) is bounded but not convergent and, thus, the converse of Theorem 19 is not true.

**Theorem 20**

Suppose \( x_n \to x \) and \( y_n \to y \).

1. \( x_n + y_n \to x + y \)
2. \( x_n y_n \to x y \)

3. If \( c \in \mathbb{R} \), then \( c x_n \to c x \).

4. If \( x_n, x \neq 0 \), then \( \frac{1}{x_n} \to \frac{1}{x} \).

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**Proof**

We leave parts 1-3 as exercises. Let \( \varepsilon > 0 \). Since

i. \( \{x_n\}_{n=1}^{\infty} \) is bounded (Proposition 19)

ii. \( \mathbb{R} \) satisfies the *greatest lower bound property*,

iii. \( x_n \neq 0 \),

there exists a real number \( L \) so that \( 0 < L \leq |x_n| \) for all \( n \in \mathbb{N} \). Because \( x_n \to x \), there exists \( N \) so that if \( n \geq N \), \( |x_n - x| < \varepsilon L |x| \). For \( n \geq N \) we have

\[
\left| \frac{1}{x_n} - \frac{1}{x} \right| = \left| \frac{x - x_n}{x_n x} \right| \leq \frac{|x - x_n|}{L |x|} < \varepsilon.
\]

Thus, \( \frac{1}{x_n} \to \frac{1}{x} \). \( \triangle \)

**Illustration 4**

Let \( x_n = n \) and \( y_n = \frac{1}{n} - n \). Then neither sequence converges yet the sum sequence \( x_n + y_n \) is a convergent sequence.
Definition 21

A sequence \( \{x_n\}_{n=1}^\infty \) in \( \mathbb{R} \) is said to be a **Cauchy sequence in** \( \mathbb{R} \) iff for each positive real number \( \varepsilon \) there exists a natural number \( N \) so that for all indices \( n, m \geq N \), \( |x_n - x_m| < \varepsilon \).

A sequence in \( \mathbb{R} \) is *Cauchy* provided that given any positive “error” \( \varepsilon \) there exists a special index \( N \) (usually depending on \( \varepsilon \) - that is, the index \( N \) is a function of \( \varepsilon \)) beyond which the distance between all pairs of sequence members is strictly less than the prescribed error \( \varepsilon \). Crudely, a sequence \( \{x_n\}_{n=1}^\infty \) in \( \mathbb{R} \) is *Cauchy* provided the terms \( x_n \) and \( x_m \) are “close” when \( n \) and \( m \) are both large.

**Example 11**

1. The sequence \( \left\{ \frac{1}{n} \right\}_{n=1}^\infty \) is a Cauchy sequence.

**Proof**

Let \( \varepsilon \) be any positive rational number. Find a natural number \( N \) so that \( \frac{1}{N} < \varepsilon \). (For example, if \( \varepsilon = 0.1 \), then \( N = 11 \) or \( N = 12 \) or \( N = 13 \) or ... . If \( \varepsilon = \frac{12}{2567} \), then \( N = 214 \) or \( N = 215 \) or \( N = 216 \) or ... . We observe that \( N \) is not unique for a given value of \( \varepsilon \).) Then for \( n, m \geq N \) we have \( \left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{\min[n,m]} \leq \frac{1}{N} < \varepsilon \). Hence, the given sequence is Cauchy. (In the special case that \( \varepsilon = 0.1 \) we have just proved that for any natural numbers \( n \) & \( m \) greater than or equal to 11 that \( \left| \frac{1}{n} - \frac{1}{m} \right| < 0.1 \). We note that a general proof not depending on any specific value of \( \varepsilon \) is the desired goal here.)
Note that in the above we used the fact that the sequence $\left\{ \frac{1}{n} \right\}_{n=1}^\infty$ is convergent to establish that it is Cauchy!

2. We now define a sequence in $\mathbb{Q}$ recursively as follows: set $x_1 = \frac{1}{2}$ and for $n > 1$ set $x_{n+1} = \frac{1}{2 + x_n}$. Here, the sequence is $\left\{ \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \frac{70}{169}, \frac{169}{408}, \ldots \right\}$. This sequence is a Cauchy. (I will make no statement here concerning the convergence or divergence of the recursively defined sequence. Can you make such a statement?)

**Proof (nontrivial!)**

(A “trick” and a little calculus first!) Consider the function $f(x) = \frac{1}{2 + x}$. For $x$ in $[0,1]$, $f(x)$ is a decreasing function and has values in the interval in $[f(1), f(0)] = \left[ \frac{1}{3}, \frac{1}{2} \right]$. It follows that

$$\frac{1}{3} \leq x_n \leq \frac{1}{2} \quad \text{since} \quad x_1 = \frac{1}{2}.$$ 

By basic calculus (including the **Mean-Value Theorem**),

$$|f(u) - f(v)| \leq \frac{1}{4} |u - v| \quad \text{for} \quad u, v \in \left[ \frac{1}{3}, \frac{1}{2} \right].$$

Hence,

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq \frac{1}{4} |x_n - x_{n-1}|$$

(This condition makes the given sequence a **contractive sequence**.) WLOG, suppose $m > n$. Then

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + (x_{m-2} - x_{m-3}) + \ldots + (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + \ldots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|$$

$$\leq \frac{1}{4} |x_{m-1} - x_{m-2}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + \ldots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|$$
Now, let $\theta$ be any positive rational number. Since $\frac{2}{15}\left(\frac{1}{4}\right)^{n-1} \to 0$ as $n \to \infty$, we can find $N$ so
that \( \frac{2}{15} \left( \frac{1}{4} \right)^{N-1} < \varepsilon \). (For example, if \( \varepsilon = 0.0001 \), then the smallest \( N \) that works is \( N = 7 \).

Finding the smallest \( N \) for a given \( \varepsilon \) is always the goal! Having said that, we’re happy when we simply find an index \( N \) that works!) Hence, for \( m > n \geq N \) we have that

\[
|a_m - a_n| \leq \frac{4}{3} \left| a_{m+1} - a_n \right| \leq \frac{2}{15} \left( \frac{1}{4} \right)^{n-1} \leq \frac{2}{15} \left( \frac{1}{4} \right)^{N-1} < \varepsilon.
\]

We conclude the sequence \( \left\{ \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \frac{70}{169}, \frac{169}{408}, \ldots \right\} \) is Cauchy.

3. Suppose we have the sequence of rational numbers

\[
\left\{ 0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \ldots \right\}
\]

a. For any fixed \( k \in \mathbb{N} \), explain why \( \lim_{n \to \infty} |a_{n+k} - a_n| = 0 \).

b. Explain why the given sequence is not a Cauchy sequence. (Hint: Let \( \varepsilon < 1 \).)

**Illustration 5**

1. Prove that the sequence \( \left\{ \frac{1}{2}, \frac{2}{5}, \frac{5}{19}, \frac{19}{810}, \frac{810}{144}, \frac{144}{30}, \frac{30}{280}, \frac{280}{5760}, \ldots \right\} \) where the \( n^{\text{th}} \) term is given by \( a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \) is a Cauchy sequence.

**Proof**

We note that if \( m > n \), then
An induction argument shows that \(2^{k-1} \leq k!\) for all \(k \in \mathbb{N}\) and so it follows that \(\frac{1}{k!} \leq \frac{1}{2^{k-1}}\).

Hence,

\[
|x_\infty - x_n| = \left| \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!} - \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} \right|
= \ldots
= \left| \frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \frac{(-1)^{n+4}}{(n+3)!} + \ldots + \frac{(-1)^{n+1}}{m!} \right|
\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots + \frac{1}{m!}
\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}
\leq \frac{1}{2^{n-1}}.
\]

Now, for any positive rational number \(\varepsilon\) choose a natural number \(N\) so that \(\frac{1}{2^{N-1}} < \varepsilon\). Then for 

\(n, m > N\) we have that 

\[|x_\infty - x_n| < \frac{1}{2^{\min\{m,n\}}} \leq \frac{1}{2^{N-1}} < \varepsilon\]

and it follows that \(\left\{\frac{\sum_{k=1}^n (-1)^{k+1}}{k!}\right\}_{n=1}^\infty\) is a Cauchy sequence. \(\blacksquare\)
2. Show that the sequence \( x_n = \ln n \) is not a Cauchy sequence even though
\[
\lim_{n \to \infty} (x_{n+1} - x_n) = 0.
\]

**Proof**

From ordinary calculus, we have that
\[
\lim_{n \to \infty} (x_{n+1} - x_n) = \lim_{n \to \infty} \ln \frac{n + 1}{n} = \ln \left( \lim_{n \to \infty} \frac{n + 1}{n} \right) = \ln 1 = 0.
\]

Again, from ordinary calculus, we recall that the continuous function \( y = \ln x \) increases without bound. That is, \( \lim_{x \to \infty} \ln x = +\infty \). Now, *on the contrary*, let’s suppose that \( x_n = \ln n \) is a Cauchy sequence. Then for all positive rational numbers \( \varepsilon \) there exists a natural number \( N \) so that if \( n, m > N \), we have that \( |x_n - x_m| < \varepsilon \). Take \( \varepsilon = 1 \). Then for any fixed \( N \) there exists natural numbers \( m, n > N \) with \( N < \ln n < N^2 < N^2 + 1 < \ln m < N^3 \) since \( \lim_{x \to \infty} \ln x = +\infty \). Thus,
\[
|x_n - x_m| > 1
\]
which is a contradiction. It must be that \( x_n = \ln n \) fails to be a Cauchy sequence.

We note that the set of Cauchy sequences is closed under the operations of addition and multiplication.

**Illustration 6**

Suppose that \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) are Cauchy sequences in \( \mathbb{R} \). Then \( \{x_n\}_{n=1}^{\infty} + \{y_n\}_{n=1}^{\infty} \) and \( \{x_n\}_{n=1}^{\infty} \cdot \{y_n\}_{n=1}^{\infty} \) are also Cauchy sequences in \( \mathbb{R} \).

**Proof - Addition Case**

Let \( \varepsilon \) be any positive rational number. Since \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R} \), there exists \( N_x \)}
such that if \( m, n \geq N_x \), \( |x_m - x_n| < \frac{\varepsilon}{2} \). Similarly, since \( \{y_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \), there exists \( N_y \) such that if \( m, n \geq N_y \), \( |y_m - y_n| < \frac{\varepsilon}{2} \). Set \( N = \max \{N_x, N_y\} \). Then for \( m, n \geq N \) we have

\[
\left| (x_m + y_m) - (x_n + y_n) \right| = \left| (x_m - x_n) + (y_m - y_n) \right| \\
\leq |x_m - x_n| + |y_m - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, \( \{x_n + y_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \).

The proof that the product of Cauchy sequences is again Cauchy requires the following standard fact that all Cauchy sequences are bounded. That is, for a given Cauchy sequence we can find a pair of horizontal lines such that the graph of the sequence is contained between these two lines.

**Theorem 22**

Every Cauchy sequence in \( \mathbb{R} \) is bounded. That is, if \( \{x_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \), then there exist a positive real number \( M \) so that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \).

**Proof**

Take \( \varepsilon = 1 \). Since \( \{x_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \), there exists \( N \) such that if \( m, n \geq N \), \( |x_m - x_n| < \varepsilon = 1 \). Hence, for all \( n \geq N \) we have

\[
|x_n - x_N| < 1 \quad \Rightarrow \quad |x_n| < |x_N| + 1.
\]

Set \( M = \max \{ |x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1 \} \). It follows that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \).

How does the “working” definition of a Cauchy sequence compare with the “working” definition of a convergent sequence?
Theorem 23 - Cauchy Convergence Criterion

A sequence \( \{x_n\}_{n=1}^{\infty} \) of real numbers is Cauchy iff the sequence is convergent.

Proof

Exercise.

We say that a sequence is **monotone** iff it is either nondecreasing or nonincreasing.

**Illustration 7**

1. The sequence \( x_n = \frac{1}{n} \) is nonincreasing and, hence, monotone.

2. The sequence \( x_n = \sum_{k=0}^{n} \frac{1}{k!} \) is nondecreasing and, hence, monotone.

3. The sequence \( \left\{ 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \ldots \right\} \) is neither nondecreasing nor nonincreasing and so this sequence is *not* monotone.

Theorem 24 - Monotone Convergence Theorem

Suppose that the sequence \( \{x_n\}_{n=1}^{\infty} \) is monotone. Then \( \{x_n\}_{n=1}^{\infty} \) is converges iff \( \{x_n\}_{n=1}^{\infty} \) is bounded.

Proof

Assume \( \{x_n\}_{n=1}^{\infty} \) is monotone increasing (i.e. non decreasing). (The other case is similar.)

(\( \Rightarrow \)) Done.

(\( \Leftarrow \)) Assume \( \{x_n\}_{n=1}^{\infty} \) is bounded. Then there exists \( M \) so that \( -M \leq x_n \leq M \) for all \( n \in \mathbb{N} \). By the *least upper bound property*, there exist a number \( x \) such that \( x = \text{lub} \ \{x_1, x_2, x_3, \ldots \} \). Note:
\( x_n \leq x \) for all \( n \in \mathbb{N} \). Claim: \( x_n \to x \). On the contrary, suppose not. Then there exists some \( \varepsilon > 0 \) so that \( |x_n - x| \geq \varepsilon \) for all \( n \in \mathbb{N} \). It follows that \( x - x_n > \varepsilon \) or, equivalently, \( x - \varepsilon > x_n \) for all \( n \in \mathbb{N} \). Thus, \( x - \varepsilon \) is an upper bound for the set \( \{x_1, x_2, x_3, \ldots\} \) smaller than the least upper bound. A contradiction. We conclude that \( x_n \to x \). ∗

The power of the Monotone Convergence Theorem lies in that fact that one doesn’t need to know the limiting value for \( \{x_n\}_{n=1}^\infty \) to establish the convergence of the sequence.

**Example 12**

Set \( x_1 = 2 \) and \( x_{n+1} = \sqrt{2x_n + 3} \). Prove that the sequence \( \{x_n\}_{n=1}^\infty \) converges by the Monotone Convergence Theorem.

**Proof**

We first prove that \( x_n < 3 \) for all \( n \). The result is clearly true for \( n = 1 \). So, suppose it is valid for \( k \) and prove it is valid for \( k + 1 \). Since \( x_k < 3 \), it follows that

\[
x_{k+1} = \sqrt{2x_k + 3} < \sqrt{2(3) + 3} = \sqrt{9} = 3.
\]

Since the function \( f(x) = \sqrt{2x + 3} - x > 0 \) for \( x \in [2, 3] \), it follows that \( \sqrt{2x + 3} > x \) there and so \( x_{n+1} = \sqrt{2x_n + 3} > x_n \) for all natural numbers \( n \). Thus, the given sequence is a bounded, monotone sequence and therefore it converges by the Monotone Convergence Theorem. (The limiting values happens to be 3. Prove it! Hint: Solve \( L = \sqrt{2L + 3} \).) ∗

An increasing sequence either converges or diverges to positive infinity. Similarly, a decreasing sequence either converges or diverges to negative infinity.
Illustration 8

Show that \( e_n = \left(1 + \frac{1}{n}\right)^n \) is a convergent sequence.

Solution (Sketch)

We plan to use the Monotone Convergence Theorem here. Applying the Binomial Theorem to \( e_n = \left(1 + \frac{1}{n}\right)^n \) we obtain:

\[
e_s = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1 \cdot n} + \frac{n(n-1)}{2! \cdot n^2} + \frac{n(n-1)(n-2)}{3! \cdot n^3} + \ldots + \frac{n(n-1) \ldots (2)(1)}{n! \cdot n^n}
\]

By applying the above process to obtain a representation for \( e_{n+1} \), we observe that each term in \( e_s \) is less than or equal to the corresponding term in \( e_{n+1} \) and \( e_{n+1} \) has one more positive term than \( e_s \). So, \( e_s \leq e_{n+1} \) and we conclude that \( e_s = \left(1 + \frac{1}{n}\right)^n \) is a nondecreasing sequence. Now, we show that the given sequence is bounded. First off, by mathematical induction, we have that \( k! \geq 2^{k-1} \) and so \( \frac{1}{k!} \leq \frac{1}{2^{k-1}} \). From the above work we have
\[ e_n = \left( 1 + \frac{1}{n} \right)^n \]

\[ = 1 + 1 + \frac{1}{2!} \left( 1 - \frac{1}{n} \right) + \frac{1}{3!} \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) + \ldots + \frac{1}{n!} \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \ldots \left( 1 - \frac{n}{n} \right) \]

\[ \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \]

\[ \leq 1 + \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n+1}} \right) \]

\[ \leq 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \]

\[ \leq 3. \]

By the *Monotone Convergence Theorem*, the bounded, monotone sequence \[ e_n = \left( 1 + \frac{1}{n} \right)^n \]

converges. We note that the MCT does not provide us with the limit for this particular sequence.

With additional effort one can show that \[ \left( 1 + \frac{1}{n} \right)^n - e \approx 2.71828. \]